

DOUBLE AFFINE HECKE ALGEBRA IN LOGARITHMIC CONFORMAL FIELD THEORY

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ABSTRACT. We construct the representation of Double Affine Hecke Algebra whose symmetrization gives the center of the quantum group $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ and by Kazhdan–Lusztig duality the Verlinde algebra of $(1, p)$ models of logarithmic conformal field theory.

1. INTRODUCTION

Recently, quantum group methods led (see the recent review [1]) to a progress in logarithmic conformal field theory [2]. For the $(1, p)$ models [3], an equivalence between representation categories of the chiral algebra and the quantum group $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ was established [4] in the general framework of the Kazhdan–Lusztig duality [5]. Remarkably, the KL duality extends to an isomorphism between modular group representations on the quantum group center and on the space of generalized characters [6] of a $(1, p)$ model. Moreover, the Verlinde algebra of $(1, p)$ models [7, 8] (see also [9]) coincides [6] with the Grothendieck ring of $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$.

An unusual property of logarithmic conformal field theory is the nonsemisimplicity of the Verlinde algebra. However, this phenomenon does not look extraordinary in the Double affine Hecke algebra representation framework [11] of the Verlinde algebra classification. It leads to a natural conjecture [12] that the $(1, p)$ model Verlinde algebra can be realized in terms of a DAHA representation. Indeed, the representation of DAHA whose symmetrization gives the center of the quantum group $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ and therefore the Verlinde algebra of $(1, p)$ models is identified in the present paper.

1.1. DAHA. We consider the symplectic DAHA [11] generated by X , Y , and T with the relations

$$(1.1) \quad TXT = X^{-1}, \quad TY^{-1}T = Y, \quad XY = \mathfrak{q}YXT^2,$$

$$(1.2) \quad (T - \mathfrak{t}^{\frac{1}{2}})(T + \mathfrak{t}^{-\frac{1}{2}}) = 0, \quad \mathfrak{t} = \mathfrak{q}^2.$$

In the paper we fix the deformation parameter

$$(1.3) \quad \mathfrak{q} = e^{\frac{i\pi}{p}}, \quad \mathfrak{q}^{\frac{1}{2}} = e^{\frac{i\pi}{2p}},$$

where $p = 3, 4, 5, 6, \dots$. We let \mathcal{H} denote this algebra. The group $PSL(2, \mathbb{Z})$ acts by automorphisms on \mathcal{H}

$$(1.4) \quad \tau_+ : \quad Y \rightarrow \mathfrak{q}^{-1/2}XY, \quad X \rightarrow X, \quad T \rightarrow T$$

$$(1.5) \quad \tau_- : \quad X \rightarrow \mathfrak{q}^{1/2} Y X, \quad Y \rightarrow Y, \quad T \rightarrow T$$

where

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \tau_+, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \tau_-.$$

We note that the Fourier transform is given by

$$(1.6) \quad \sigma : \quad X \rightarrow Y^{-1}, \quad Y \rightarrow X T^2, \quad T \rightarrow T, \\ \sigma = \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}.$$

1.2. The representation. We consider a $6p - 4$ -dimensional reducible but indecomposable representation \mathcal{Z} of \mathcal{H} . The representation \mathcal{Z} contains the maximal subrepresentation V^{-2} , which in notations of [11] is defined as the quotient $V^{-2} = \mathcal{P}/(X^{2p} + X^{-2p} - 2)$, where $\mathcal{P} = \mathbb{C}[X, X^{-1}]$ is the standard representation of \mathcal{H} in the Lourent polynomials. The $2p - 4$ -dimensional irreducible quotient $\mathcal{M} = \mathcal{Z}/V^{-2}$ is isomorphic to the representation V_{2p-4} from [11] given by the quotient $\mathcal{P}/\varepsilon_{-p+2}$, where $\varepsilon_{-p+2} = \prod_{j=2}^{p-1} (\mathfrak{q}^{-j} X - \mathfrak{q}^j X^{-1})$. We note also that V^{-2} is also reducible and contains the maximal $2p + 4$ -dimensional irreducible subrepresentation \mathcal{W} and the quotient $\mathcal{E} = V^{-2}/\mathcal{W}$ is isomorphic to \mathcal{M} .

In Sec. 2, we describe the representation \mathcal{Z} by the explicit action of operators T , X and Y in a basis. Then we describe its structure and explicitly find the subrepresentation and quotients.

The representation \mathcal{Z} bears a commutative associative multiplication, which is described in Sec. 1.5. The multiplication gives further the multiplication in the Verlinde algebra.

The $PSL(2, \mathbb{Z})$ generators σ and τ_+ are realized as a conjugation with some operators \mathcal{S} and \mathfrak{v} respectively, acting in \mathcal{Z} . The operator \mathfrak{v} acts by a multiplication with an element from \mathcal{Z} , which is abusing notation denoted by the same symbol \mathfrak{v} . By analogy with [11] we call \mathfrak{v} the Gaussian element.

1.3. Symmetrization. The operator T has two different eigenvalues \mathfrak{q} and $-\mathfrak{q}^{-1}$ in \mathcal{Z} . The eigenspace of T with the eigenvalue \mathfrak{q} is $3p - 1$ -dimensional. We let $\mathcal{T}_{\mathfrak{q}}$ denote this eigenspace. In accordance with the general theory [11], $\mathcal{T}_{\mathfrak{q}}$ is an associative commutative algebra with multiplication induced by the multiplication in \mathcal{Z} and at the same time a representation of $SL(2, \mathbb{Z})$ induced by the $PSL(2, \mathbb{Z})$ -action in \mathcal{Z} . The operators \mathcal{S} , \mathfrak{v} , $C = -(X + X^{-1})$ and $H = -(Y + Y^{-1})$ have well defined restrictions on $\mathcal{T}_{\mathfrak{q}}$. Now we are ready to formulate the main result of the paper (the reader can find all needed quantum group definitions in [6]).

1.4. Theorem. • $\mathcal{T}_{\mathfrak{q}}$ is isomorphic to the center of $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ as associative commutative algebra and as $SL(2, \mathbb{Z})$ representation.

- *Under the isomorphism the eigenvectors of C correspond to Radford images and eigenvectors of H correspond to Drinfeld images of the characters of $\overline{\mathcal{U}}_{\mathfrak{q}}\mathfrak{sl}(2)$ irreducible representations.*
- *The Gaussian element v coincides with the ribbon element of $\overline{\mathcal{U}}_{\mathfrak{q}}\mathfrak{sl}(2)$.*

In Sec. 4, we describe the subspace of T with the eigenvalue \mathfrak{q} , and in Sec. 5 we give the proof of theorem 1.4. The notations in this part correspond to the same notations in [6].

1.5. Structure of \mathcal{Z} . The very important information about the representation \mathcal{Z} is encoded in the spectra of operators X and Y . These operators are not diagonalizable but both have Jordan blocks of dimension 2. In order to describe their Jordan structure we introduce two bases in which operators X and Y^{-1} have a Jordan form. We call the first basis the X -basis and the second one the Y -basis. Jordan forms of X and Y^{-1} coincide in \mathcal{Z} .

1.5.1. X -basis. The representation \mathcal{Z} has the basis

(1.7)

$$w_1 \dots w_{2p}, e_1, e_p, e_{p+1}, e_{2p}; \quad e_2 \dots e_{p-1}, e_{p+2} \dots e_{2p-1}; \quad m_2 \dots m_{p-1}, m_{p+2} \dots m_{2p-1}.$$

The subrepresentation \mathcal{W} is spanned by elements $w_1 \dots w_{2p}, e_1, e_p, e_{p+1}$ and e_{2p} . The elements $e_2 \dots e_{p-1}, e_{p+2} \dots e_{2p-1}$ ($m_2 \dots m_{p-1}, m_{p+2} \dots m_{2p-1}$) give a basis in \mathcal{E} (in \mathcal{M}) under the canonical projection. In basis (1.7) we have

$$(1.8) \quad Xw_s = \mathfrak{q}^s w_s, \quad Xe_s = \mathfrak{q}^s(e_s + w_s), \quad Xm_s = \mathfrak{q}^s m_s.$$

We call (1.7) the X -basis.

1.5.2. The multiplication in \mathcal{Z} . The representation \mathcal{Z} is endowed with a commutative associative multiplication, which is naturally written in basis (1.7) as

$$(1.9) \quad e_i e_j = \delta_{i,j} e_j, \quad e_i w_j = \delta_{i,j} w_j, \quad e_i m_j = \delta_{i,j} m_j, \quad w_i w_j = w_i m_j = m_i m_j = 0.$$

1.5.3. Y -basis. The representation \mathcal{Z} contains the basis

(1.10)

$$u_1 \dots u_{2p}, f_1, f_p, f_{p+1}, f_{2p}; \quad f_2 \dots f_{p-1}, f_{p+2} \dots f_{2p-1}; \quad k_2 \dots k_{p-1}, k_{p+2} \dots k_{2p-1}$$

in which Y^{-1} acts as follows

$$(1.11) \quad Y^{-1}u_s = \mathfrak{q}^s u_s, \quad Y^{-1}f_s = \mathfrak{q}^s(f_s + u_s), \quad Y^{-1}k_s = \mathfrak{q}^s k_s.$$

We call (1.10) the Y -basis.

In Subsec. 2.3, we find the Y -basis and give decompositions of elements from the Y -basis in the X -basis.

1.5.4. $PSL(2, \mathbb{Z})$ -action. The operator \mathcal{S} maps the X -basis to the Y -basis

$$(1.12) \quad \mathcal{S}w_s = u_s, \quad \mathcal{S}e_s = f_s, \quad \mathcal{S}m_s = k_s.$$

In Subsec. 3.1, we establish properties of the \mathcal{S} -operator. By a direct calculation, using the decompositions of the Y -basis in the X -basis, we check that this operator satisfies all relations (1.6) and $\mathcal{S}^2 = \mathbf{q}T^{-1}$.

In terms of the X -basis, the Gaussian element is

$$(1.13) \quad v = \sum_{s=1}^{2p} \mathbf{q}^{-\frac{1}{2}(s^2-1)} e_s - w_1 + \mathbf{q}^{-\frac{p^2}{2}} w_{p+1} + \left(\sum_{s=2}^{p-1} + \sum_{s=p+2}^{2p-1} \right) \mathbf{q}^{-\frac{1}{2}(s^2-1)} ((p-s)w_s + pm_s).$$

The properties of this element are described in Subsec. 3.2.

In the end of Sec. 3, we prove the $PSL(2, \mathbb{Z})$ relations.

1.6. Notation. We introduce Chebyshov polynomials

$$(1.14) \quad U_s(x) = x^{s-1} + x^{s-3} + \dots + x^{-(s-3)} + x^{-(s-1)}.$$

In what follows we often use the numbers

$$(1.15) \quad \{s\} = \frac{\mathbf{q}^s + \mathbf{q}^{-s}}{\mathbf{q} - \mathbf{q}^{-1}}, \quad [s] = \frac{\mathbf{q}^s - \mathbf{q}^{-s}}{\mathbf{q} - \mathbf{q}^{-1}},$$

$$(1.16) \quad \omega_s = \frac{p\sqrt{2p}}{[s]^2} (-1)^{p+s+1}, \quad \xi_s \equiv \frac{-(-1)^{p-s} p\sqrt{2p}}{\mathbf{q}^s - \mathbf{q}^{-s}},$$

$$(1.17) \quad [s, j] \equiv \begin{cases} s, & j = 0, 2p, \\ (-1)^{s-1} s, & j = p, \\ \frac{[sj]}{[j]}, & j \bmod p \neq 0, \end{cases} \quad \{s, j\} \equiv \begin{cases} 0, & j \bmod p = 0, \\ \frac{\{sj\}}{[j]}, & \text{otherwise.} \end{cases}$$

2. REPRESENTATION

In this section, we recall the representation V^{-2} [11] and then define the representation \mathcal{Z} , which is an extension of V^{-2} . Then we find a Jordan basis for Y in which Y^{-1} acts by (1.11).

2.1. Polynomial representation V^{-2} . The representation \mathcal{Z} is an extension of the representation V^{-2} from [11]. To describe V^{-2} we recall the standard representation [11] of \mathcal{H} in the space of Laurent polynomials $\mathcal{P} = \mathbb{C}[X, X^{-1}]$. The \mathcal{H} generators act as follows

$$(2.1) \quad T \rightarrow \mathbf{t}^{\frac{1}{2}} s + \frac{\mathbf{t}^{\frac{1}{2}} - \mathbf{t}^{-\frac{1}{2}}}{X^2 - 1} (s-1), \quad \mathbf{t} = \mathbf{q}^k,$$

$$(2.2) \quad Y \rightarrow -s p T,$$

where

$$(2.3) \quad \mathfrak{s}f(X) = f(X^{-1}), \quad \mathfrak{p}f(X) = f(\mathfrak{q}X)$$

and X, X^{-1} act by multiplication. (We note that these formulas differ from [11] by the sign in the definition of $Y \rightarrow \mathfrak{s} \mathfrak{p} T$.) The representation V^{-2} is the $4p$ -dimensional representation in the quotient space $\mathcal{P}/(X^{2p} + X^{-2p} - 2)$.

2.1.1. Proposition. • *The operators X and Y have in V^{-2} eigenvalues $\mathfrak{q}^s, s = 1 \dots 2p$, each with multiplicity 2.*

- *The Jordan basis of X contains functions e_s and w_s for $s = 1 \dots 2p$.*
- *The Jordan basis of Y contains functions u_s for $s = 1 \dots 2p$ and k_s for $s = 2 \dots p-1, p+2 \dots 2p-1$, and functions $f_1, f_p, f_{p+1}, f_{2p}$.*

The action of X and Y^{-1} in these bases is

$$(2.4) \quad Xw_s = \mathfrak{q}^s w_s, \quad Xe_s = \mathfrak{q}^s (e_s + w_s),$$

$$(2.5) \quad Y^{-1}u_s = \mathfrak{q}^s u_s, \quad Y^{-1}f_s = \mathfrak{q}^s (f_s + u_s), \quad Y^{-1}k_s = \mathfrak{q}^s k_s.$$

Proof. To describe the spectra of these operators, we introduce functions

$$(2.6) \quad \begin{aligned} w_s &= \frac{1}{4p^2} (X^{2p} - 1) \sum_{j=0}^{2p-1} \mathfrak{q}^{-sj} X^j, \\ e_s &= \frac{1}{2p} + \frac{1}{4p^2} \sum_{j=1}^{2p-1} (2p-j)(\mathfrak{q}^{-sj} X^j + \mathfrak{q}^{sj} X^{-j}), \end{aligned} \quad s = 1 \dots 2p,$$

and

$$(2.7) \quad \begin{aligned} u_s &= \frac{(-1)^s}{p\sqrt{2p}} \left(\mathfrak{q}^s \frac{U_{p-s}(X) + U_{p+s}(X)}{2} + \mathfrak{q} \frac{U_{p-s}(\mathfrak{q}^{-1}X) + U_{p+s}(\mathfrak{q}^{-1}X)}{2} \right), \\ u_{p+s} &= \frac{(-1)^{p+s}}{p\sqrt{2p}} \left(\mathfrak{q}^{p+s} \frac{U_s(X) + U_{2p-s}(X)}{2} + \mathfrak{q} \frac{U_s(\mathfrak{q}^{-1}X) + U_{2p-s}(\mathfrak{q}^{-1}X)}{2} \right), \end{aligned} \quad s = 1 \dots p,$$

$$(2.8) \quad \begin{aligned} k_s &= \frac{(-1)^{s+1}}{p\sqrt{2p}} (\mathfrak{q}^s U_{p-s}(X) + \mathfrak{q} U_{p-s}(\mathfrak{q}^{-1}X)), \\ k_{p+s} &= \frac{(-1)^{p+s}}{p\sqrt{2p}} (\mathfrak{q}^{p+s} U_s(X) + \mathfrak{q} U_s(\mathfrak{q}^{-1}X)), \end{aligned} \quad s = 2 \dots p-1$$

$$(2.9) \quad f_s = \frac{1}{p\sqrt{2p}} \cdot \begin{cases} \frac{(-1)^{p+1} U_{2p}(X)}{2}, & s = p, \\ U_p(X), & s = 2p, \\ \frac{(-1)^p \mathfrak{q} X U_{2p}(X)}{2}, & s = p+1, \\ -q X U_p(X), & s = 1. \end{cases}$$

Then, (2.4) is checked by a direct calculation. It is easy to see that $1 = \sum_{s=1}^{2p} e_s$. Together with (2.4) this gives

$$X^j = \sum_{s=1}^{2p} q^{sj} (e_s + jw_s), j = 0, \pm 1, \pm 2 \dots$$

i.e. functions e_s and w_s are linearly independent and form a basis in V^{-2} .

(2.5) is also checked by a direct calculation using the following relations

$$\begin{aligned} YU_s(q^{-1}X) &= (q^s + q^{-s})U_s(q^{-1}X) - q^{-1}U_s(X), \\ YU_s(X) &= qU_s(q^{-1}X). \end{aligned}$$

The linear independence of these vectors is proved by the standard technic (See definition 2.5.4 and theorems 2.5.9 and 2.9.3 from [11]). \square

The representation V^{-2} is reducible. It has a $2p + 4$ -dimensional subrepresentation \mathcal{W} spanned by functions $w_1 \dots w_{2p}$, e_1 , e_p , e_{p+1} and e_{2p} . The quotient $\mathcal{E} = V^{-2}/\mathcal{W}$ is isomorphic to V_{2p-4} from [11]. The representation V_{2p-4} is defined in [11] as the quotient $V_{2p-4} = \mathcal{P}/\varepsilon_{-p+2}$, where $\varepsilon_{-p+2} = \prod_{j=2}^{p-1} (q^{-j}X - q^jX^{-1})$.

Decomposition of any polynomial $f(X)$ in the basis e_s , w_s is given by

$$(2.10) \quad f(X) = \sum_{s=1}^{2p} \left(f(q^s)e_s + \left(X \frac{df(X)}{dX} \Big|_{X=q^s} \right) w_s \right).$$

Using it, we check that $u_1 \dots u_{2p}$, f_1 , f_p , f_{p+1} and f_{2p} belong to \mathcal{W} and therefore in \mathcal{W} Jordan forms of X and Y^{-1} coincide. But in the whole V^{-2} they do not coincide, hence automorphism (1.6) cannot be realized as a conjugation. To recover this, we find an extension of V^{-2} to a $6p - 4$ -dimensional representation \mathcal{Z} by adding vectors m_2, \dots, m_{p-1} and m_{p+2}, \dots, m_{2p-1} . The action of X on them is $Xm_s = q^s m_s$. The whole \mathcal{Z} cannot be realized in a space of polynomials in 1 variable. We describe \mathcal{Z} in terms of an abstract vector space.

2.2. The representation \mathcal{Z} in the X -basis. We assume the following definition of \mathcal{Z} . The representation \mathcal{Z} is a $6p - 4$ -dimensional vector space with the basis consisting of $4p$ vectors e_s and w_s with $s = 1 \dots 2p$, and $2p - 4$ vectors m_s with $s = 2 \dots p - 1, p + 2 \dots 2p - 1$. The action of \mathcal{H} -operators in this basis is defined by the formulas:

$$(2.11) \quad Xw_s = q^s w_s, \quad Xe_s = q^s (e_s + w_s), \quad Xm_s = q^s m_s,$$

$$(2.12) \quad Tw_p = -q^{-1}w_p - (q - q^{-1})e_p, \quad Tw_{2p} = -q^{-1}w_{2p} - (q - q^{-1})e_{2p},$$

$$(2.13) \quad Tw_s = -\frac{q^{-s}}{[s]}w_s - \frac{[s-1]}{[s]}w_{2p-s}, \quad s \neq 0, p,$$

$$(2.14) \quad Te_p = qe_p, \quad Te_{2p} = qe_{2p},$$

$$(2.15) \quad Te_s = -\frac{\mathbf{q}^{-s}}{[s]}e_s + \frac{[s-1]}{[s]}e_{2p-s} + \frac{2}{(q-q^{-1})[s]^2}(w_s - w_{2p-s}), \quad s \neq 0, p,$$

$$(2.16) \quad Tm_{2p-1} = \mathbf{q}m_{2p-1} - (\mathbf{q} + \mathbf{q}^{-1})w_1,$$

$$(2.17) \quad Tm_{p-1} = \mathbf{q}m_{p-1},$$

$$(2.18) \quad Tm_s = \frac{-\mathbf{q}^{-s}}{[s]}m_s + \frac{[s-1]}{[s]}m_{2p-s} \quad s = 2 \dots p-2, p+2 \dots 2p-2,$$

$$(2.19) \quad Yw_p = -q^{-1}w_{p+1} + (q - q^{-1})e_{p+1}, \quad Yw_{2p} = -q^{-1}w_1 + (q - q^{-1})e_1,$$

$$(2.20) \quad Yw_s = -\frac{\mathbf{q}^{-s}}{[s]}w_{2p-s+1} - \frac{[s-1]}{[s]}w_{s+1}, \quad s \neq 0, p,$$

$$(2.21) \quad Ye_p = -\mathbf{q}e_{p+1}, \quad Ye_{2p} = -\mathbf{q}e_1,$$

$$(2.22)$$

$$Ye_s = \frac{\mathbf{q}^{-s}}{[s]}e_{2p-s+1} - \frac{[s-1]}{[s]}e_{s+1} - \frac{2}{(q-q^{-1})[s]^2}(w_{s+1} - w_{2p-s+1}), \quad s \neq 0, p,$$

$$(2.23) \quad Ym_{2p-1} = -\mathbf{q}m_2 - (\mathbf{q} + \mathbf{q}^{-1})w_0, \quad Ym_{p-1} = -\mathbf{q}m_{p+2},$$

$$(2.24) \quad Ym_s = -\frac{[s-1]}{[s]}m_{s+1} + \frac{\mathbf{q}^{-s}}{[s]}m_{2p-s+1}, \quad s = 2 \dots p-2, p+2 \dots 2p-2.$$

We note that the basis e, w, m by definition is the X -basis (1.7) and (2.11) gives the Jordan structure of X .

2.2.1. Lemma. *Operators X, Y and T defined by (2.11)–(2.24) satisfy the DAHA relations (1.1) and (1.2).*

Proof. A direct calculation. □

We define a commutative associative multiplication in \mathbb{Z} by formulas (1.9).

2.2.2. Proposition. *\mathbb{Z} is reducible. The $2p+4$ -dimensional subspace*

$$(2.25) \quad \mathcal{W} \equiv \{w_1 \dots w_{2p}, e_1, e_p, e_{p+1}, e_{2p}\}$$

is invariant under the \mathcal{H} -action and is therefore a subrepresentation. The quotient is a direct sum: $\mathbb{Z}/\mathcal{W} = \mathcal{E} \oplus \mathcal{M}$, where $\mathcal{E} \equiv \{e_2 \dots e_{p-1}, e_{p+2} \dots e_{2p-1}\}$ and $\mathcal{M} \equiv \{m_2 \dots m_{p-1}, m_{p+2} \dots m_{2p-1}\}$.

Proof. Immediately follows from (2.11)–(2.24). □

2.3. Y -basis. In this subsection we prove that the Jordan form of Y is (1.11).

2.3.1. Proposition. *A Jordan basis of Y consists of $6p-4$ vectors: $4p$ vectors f_s, u_s for $s = 1, \dots, 2p$, and $2p-4$ vectors k_s for $s = 2, \dots, p-1, p+2 \dots 2p-1$. The action of Y^{-1} on this vectors is given by (1.11).*

Proof. We define in the X -basis the vectors

$$(2.26) \quad u_s = \sum_{j=1}^{2p} u_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} u_{j,s}^{(e)} e_j, \quad s = 1 \dots 2p,$$

where coefficients are

$$(2.27) \quad \begin{aligned} u_{j,s}^{(w)} &= \frac{(-1)^{s+j}}{\sqrt{2p}} \left(q^s \{s, j\} - q \{s, j-1\} \right), \quad j = 1 \dots 2p; \\ u_{1,s}^{(e)} &= (-1)^s \frac{q}{\sqrt{2p}}; \quad u_{2p,s}^{(e)} = (-1)^s \frac{q^s}{\sqrt{2p}}; \\ u_{p+1,s}^{(e)} &= (-1)^{p+1} \frac{q}{\sqrt{2p}}; \quad u_{p,s}^{(e)} = (-1)^{p+1} \frac{q^s}{\sqrt{2p}}; \\ u_{j,s}^{(e)} &= 0, \quad j \neq 1, p, p+1, 2p, \end{aligned}$$

the vectors

$$(2.28) \quad k_s = \sum_{j=1}^{2p} k_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} k_{j,s}^{(e)} e_j, \quad s = 2 \dots p-1, p+2 \dots 2p-1,$$

where coefficients are

$$(2.29) \quad \begin{aligned} k_{j,s}^{(w)} &= -\frac{(p-s)}{p} u_{j,s}^{(w)} - \frac{(-1)^{s+j}}{p\sqrt{2p}} \left(q^s [s, j] \{1, j\} - q [s, j-1] \{1, j-1\} \right), \\ k_{1,s}^{(e)} &= (-1)^{s+1} \frac{(q^s [s] + q(p-s))}{p\sqrt{2p}}, \quad k_{p,s}^{(e)} = (-1)^p \frac{(q[s] + q^s(p-s))}{p\sqrt{2p}}, \\ k_{p+1,s}^{(e)} &= (-1)^p \frac{(q^s [s] + q(p-s))}{p\sqrt{2p}}, \quad k_{2p,s}^{(e)} = (-1)^{s+1} \frac{(q[s] + q^s(p-s))}{p\sqrt{2p}}, \\ k_{j,s}^{(e)} &= \frac{(-1)^{s+j}}{p\sqrt{2p}} \left(q^s [s, j] - q [s, j-1] \right) \quad j \neq 1, p, p+1, 2p, \end{aligned}$$

and the vectors

$$(2.30) \quad f_s = \sum_{j=1}^{2p} f_{j,s}^{(w)} w_j + \sum_{j=1}^{2p} f_{j,s}^{(e)} e_j + \left(\sum_{j=2}^{p-1} + \sum_{j=p+2}^{2p-1} \right) f_{j,s}^{(m)} m_j, \quad s = 1 \dots 2p,$$

where coefficients are

$$(2.31) \quad \begin{aligned} f_{1,s}^{(w)} &= \frac{2(-1)^{s+1} q^{2s}}{(q - q^{-1})\sqrt{2p}}, \quad f_{p,s}^{(w)} = \frac{q(-1)^{p+1} [s]}{\sqrt{2p}}, \quad f_{p+1,s}^{(w)} = \frac{2(-1)^p q^{2s}}{(q - q^{-1})\sqrt{2p}}, \\ f_{2p,s}^{(w)} &= \frac{q(-1)^s [s]}{\sqrt{2p}}, \\ f_{j,s}^{(w)} &= -p(p-j)k_{j,s}^{(e)} + \frac{(-1)^{s+j}}{\sqrt{2p}} \left(q [s, j-1] + q^s \{s, j\} \right), \quad j \neq 1, p, p+1, 2p, \\ f_{p,s}^{(e)} &= (-1)^{p+s+1} f_{2p,s}^{(e)} = (-1)^{p+1} \frac{q^s}{\sqrt{2p}}, \quad f_{j,s}^{(e)} = 0 \quad j \neq p, 2p, \\ f_{j,s}^{(m)} &= -p^2 k_{j,s}^{(e)}, \quad j \neq 1, p, p+1, 2p \end{aligned}$$

and the coefficient $f_{j,s}^{(m)}$ in (2.30) is 0 for $s = 1, p, p+1, 2p$.

Then, (1.11) is checked by a simple calculation using formulas (2.19)–(2.24).

The linear independence of these vectors is proved in the following way. From the decompositions in the X -basis, we obtain that vectors $u_s, f_1, f_p, f_{p+1}, f_{2p}$ belong to \mathcal{W} , vectors k_s belong to $\mathcal{W} + \mathcal{E}$, and vectors f_s with $s = 2 \dots p-1, p+2 \dots 2p-1$ belong to $\mathcal{W} + \mathcal{M}$. We recall, that under the isomorphism $\mathcal{W} + \mathcal{E} \sim V^{-2}$ vectors $u_s, k_s, f_1, f_p, f_{p+1}, f_{2p}$ correspond to the linearly independent functions (2.7)–(2.9) in V^{-2} , and therefore the vectors are also linearly independent. In particular, vectors $u_s, f_1, f_p, f_{p+1}, f_{2p}$ form a basis in \mathcal{W} and therefore images of the vectors k_s under the canonical projection to $\mathcal{E} = (\mathcal{W} + \mathcal{E})/\mathcal{W}$ form a basis in \mathcal{E} . We let abusing notations k_s denote these images. We recall that the isomorphism $\mathcal{E} \sim \mathcal{M}$ maps vectors k_s (images under the canonical projections of $k_s \in \mathcal{Z}$) to f_s (images under the canonical projections of $f_s \in \mathcal{Z}$), and therefore f_s with $s = 2 \dots p-1, p+2 \dots 2p-1$ are linearly independent. Thus, the linear independence of all vectors u, f, k is established. \square

3. $PSL(2, \mathbb{Z})$ ACTION IN \mathcal{Z}

In this section we define operators \mathcal{S} and \mathbf{v} and prove that they satisfy $PSL(2, \mathbb{Z})$ relations. Conjugations with operators \mathcal{S} and \mathbf{v} give automorphisms σ and τ_+ respectively.

3.1. σ . We define the \mathcal{S} -operator that maps the X -basis to the Y -basis by formulas (1.12).

3.1.1. Proposition. \mathcal{S} satisfies relations

$$(3.1) \quad \mathcal{S}X\mathcal{S}^{-1} = Y^{-1},$$

$$(3.2) \quad \mathcal{S}Y\mathcal{S}^{-1} = XT^2,$$

$$(3.3) \quad \mathcal{S}T\mathcal{S}^{-1} = T,$$

$$(3.4) \quad \mathcal{S}^2 = \mathbf{q}T^{-1}.$$

Proof. • (3.1) follows from the definition of \mathcal{S} .

- (3.4) follows from a direct calculation of $T\mathcal{S}^2$ -action in the X -basis. We give a detailed calculation of $T\mathcal{S}^2e_s$. The calculation of $T\mathcal{S}^2w_s$ and $T\mathcal{S}^2m_s$ is similar and is omitted. We check that $T\mathcal{S}^2e_s = \mathbf{q}e_s$. We begin with

$$(3.5) \quad \begin{aligned} \mathcal{S}^2e_s &= \mathcal{S}f_s = \mathcal{S} \left(\sum_{r=1}^{2p} f_{r,s}^{(w)} w_r + f_{p,s}^{(e)} e_p + f_{2p,s}^{(e)} e_{2p} + \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) f_{r,s}^{(m)} m_r \right) = \\ &= f_{1,s}^{(w)} u_1 + f_{p,s}^{(w)} u_p + f_{p+1,s}^{(w)} u_{p+1} + f_{2p,s}^{(w)} u_{2p} + f_{p,s}^{(e)} f_p + f_{2p,s}^{(e)} f_{2p} + \\ &\quad + \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) (f_{r,s}^{(w)} u_r + f_{r,s}^{(m)} k_r). \end{aligned}$$

Then we calculate coefficients in front of e_j and w_j in (3.5) using (2.26)-(2.31). This calculation is cumbersome and is given in Appendix A. The result of the calculation is

$$(3.6) \quad \begin{aligned} \mathcal{S}^2 e_s &= -\frac{\mathfrak{q}^{s+1}}{[s]} e_s + \mathfrak{q} \frac{[s-1]}{[s]} e_{2p-s} + \frac{2(\mathfrak{q}^2 - 1)}{(\mathfrak{q}^s - \mathfrak{q}^{-s})^2} (w_s - w_{2p-s}), \quad s \neq p, 2p, \\ \mathcal{S}^2 e_p &= e_p, \quad \mathcal{S}^2 e_{2p} = e_{2p}. \end{aligned}$$

A simple calculation using (2.12)-(2.15) gives

$$T\mathcal{S}^2 e_s = \mathfrak{q} e_s, \quad s = 1 \dots 2p.$$

- (3.3) is checked as follows $\mathcal{S}^2 = \mathfrak{q}T^{-1} \Rightarrow \mathcal{S}T = T\mathcal{S} (= \mathfrak{q}\mathcal{S}^{-1}) \Rightarrow \mathcal{S}T\mathcal{S}^{-1} = T$.
- (3.2) is checked as follows $\mathcal{S}X\mathcal{S}^{-1} \stackrel{(3.1)}{=} Y^{-1} \Rightarrow \mathcal{S}X^{-1}\mathcal{S}^{-1} = Y \Rightarrow \mathcal{S}Y\mathcal{S}^{-1} = \mathcal{S}^2 X^{-1} \mathcal{S}^{-2} \stackrel{(3.4)}{=} T^{-1} X^{-1} T \stackrel{(1.1)}{=} XT^2$.

□

3.2. τ_+ . The automorphism τ_+ can be realized as a conjugation with the element $\mathbf{v} \in \mathbb{Z}$ given by (1.13).

3.2.1. Proposition. *For \mathbf{v} given by (1.13), the operator*

$$\tau_+(x) = \mathbf{v}^{-1} x \mathbf{v}, \quad \forall x \in \mathcal{H}$$

satisfies relations (1.4).

Proof. A direct calculation. □

3.2.2. Proposition. *The map*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \mathcal{S}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \mathbf{v}.$$

gives a $PSL(2, \mathbb{Z})$ action in \mathbb{Z} .

Proof. Relations (3.1)–(3.4) and (1.4) are sufficient [11] to check the $PSL(2, \mathbb{Z})$ relations. □

4. EIGENSPACE OF T WITH EIGENVALUE \mathfrak{q}

In this section, we describe the representation of the symmetrized DAHA. In section 5, we prove that it is isomorphic to the centre of $\overline{\mathcal{U}}_{\mathfrak{q}} \mathfrak{sl}(2)$.

We let $\mathcal{T}_{\mathfrak{q}}$ denote the eigenspace of T with the eigenvalue \mathfrak{q} . It is $3p - 1$ dimensional. Operators X and Y have no well-defined restriction to $\mathcal{T}_{\mathfrak{q}}$ but "symmetrized" operators $C = -(X + X^{-1})$ and $H = -(Y + Y^{-1})$ have. Indeed, for a given $\mathbf{a} \in \mathcal{T}_{\mathfrak{q}}$, we have

$$(4.1) \quad T(X + X^{-1})\mathbf{a} \stackrel{(1.1)}{=} (X^{-1}T^{-1} + TX^{-1})\mathbf{a} = (\mathfrak{q}^{-1} + T)X^{-1}\mathbf{a} \stackrel{(1.2)}{=}$$

$$= (\mathfrak{q} + T^{-1})X^{-1}\mathbf{a} = (\mathfrak{q}X^{-1} + XT)\mathbf{a} = \mathfrak{q}(X + X^{-1})\mathbf{a}.$$

Thus, $(X + X^{-1})\mathbf{a} \in \mathcal{T}_{\mathfrak{q}}$ and a similar calculation shows that H has the well-defined restriction to $\mathcal{T}_{\mathfrak{q}}$ as well.

4.1. C -basis. The eigenvectors of $C = -(X + X^{-1})$ are

$$\begin{aligned} \mathbf{e}_0 &= e_p, & \mathbf{e}_p &= e_{2p}, & \mathbf{e}_s &= e_{p+s} + e_{p-s}, \\ \mathbf{w}_1^+ &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} m_{p-1}, & \mathbf{w}_1^- &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (w_{p+1} - w_{p-1} - m_{p-1}), \\ \mathbf{w}_s^+ &= \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (m_{p-s} + m_{p+s}), & \mathbf{w}_s^- &= \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (w_{p+s} - w_{p-s} - m_{p-s} - m_{p+s}), \\ \mathbf{w}_{p-1}^+ &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (m_{2p-1} - w_1), & \mathbf{w}_{p-1}^- &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (w_{2p-1} - m_{2p-1}), \\ \mathbf{w}_s &= \mathbf{w}_s^+ + \mathbf{w}_s^- = \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (w_{p+s} - w_{p-s}), & s &= 1 \dots p-1. \end{aligned}$$

The action of C on them follows from (2.11)

$$(4.2) \quad C\mathbf{e}_0 = \mu_0\mathbf{e}_0, \quad C\mathbf{e}_p = \mu_p\mathbf{e}_p,$$

$$(4.3) \quad C\mathbf{e}_s = \mu_s\mathbf{e}_s + (\mathfrak{q} - \mathfrak{q}^{-1})^2\mathbf{w}_s, \quad s = 1 \dots p-1$$

$$(4.4) \quad C\mathbf{w}_s^{\pm} = \mu_s\mathbf{w}_s^{\pm}, \quad s = 1 \dots p-1,$$

where

$$(4.5) \quad \mu_s = \mathfrak{q}^s + \mathfrak{q}^{-s}, \quad 0 \leq s \leq p.$$

The multiplication in $\mathcal{T}_{\mathfrak{q}}$ is induced by (1.9)

$$(4.6) \quad \mathbf{e}_r\mathbf{w}_s^{\pm} = \delta_{r,s}\mathbf{w}_s^{\pm}, \quad \mathbf{e}_r\mathbf{e}_s = \delta_{r,s}\mathbf{e}_s, \quad \mathbf{w}_r^{\pm}\mathbf{w}_s^{\pm} = 0.$$

4.2. H -basis. The eigenvectors of $H = -(Y + Y^{-1})$ are

$$\begin{aligned} \mathbf{f}_0 &= f_p, & \mathbf{f}_p &= f_0, & \mathbf{f}_s &= f_{p+s} + f_{p-s}, \\ \mathbf{u}_1^+ &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} k_{p-1}, & \mathbf{u}_1^- &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (u_{p+1} - u_{p-1} - k_{p-1}), \\ \mathbf{u}_s^+ &= \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (k_{p-s} + k_{p+s}), & \mathbf{u}_s^- &= \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (u_{p+s} - u_{p-s} - k_{p-s} - k_{p+s}), \\ \mathbf{u}_{p-1}^+ &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (k_{2p-1} - u_1), & \mathbf{u}_{p-1}^- &= \frac{1}{\mathfrak{q} - \mathfrak{q}^{-1}} (u_{2p-1} - k_{2p-1}), \\ \mathbf{u}_s &= \mathbf{u}_s^+ + \mathbf{u}_s^- = \frac{[s]}{\mathfrak{q} - \mathfrak{q}^{-1}} (u_{p+s} - u_{p-s}), & s &= 1 \dots p-1. \end{aligned}$$

The action of H on them follows from (1.11)

$$(4.7) \quad H\mathbf{f}_0 = \mu_0\mathbf{f}_0, \quad H\mathbf{f}_p = \mu_p\mathbf{f}_p,$$

$$(4.8) \quad H\mathbf{f}_s = \mu_s\mathbf{f}_s + (\mathfrak{q} - \mathfrak{q}^{-1})^2\mathbf{u}_s, \quad s = 1 \dots p-1,$$

$$(4.9) \quad H\mathbf{u}_s^\pm = \mu_s \mathbf{u}_s^\pm, \quad s = 1 \dots p-1,$$

where eigenvalues are given by (4.5).

4.3. $SL(2, \mathbb{Z})$ action. Operators \mathcal{S} and \mathbf{v} have well-defined restrictions to \mathcal{T}_q . This endows \mathcal{T}_q with a representation of $SL(2, \mathbb{Z})$. In more detail, \mathcal{S} -operator in \mathcal{T}_q satisfies

$$(4.10) \quad \begin{aligned} \mathcal{S}e_s &= \mathbf{f}_s, \quad s = 0 \dots p, \\ \mathcal{S}\mathbf{w}_s^\pm &= \mathbf{u}_s^\pm, \quad s = 1 \dots p-1 \end{aligned}$$

and because $T = q$ in \mathcal{T}_q , we have $\mathcal{S}^2 = 1$. We note also that in \mathcal{T}_q relations 3.1 and 3.2 lead to

$$(4.11) \quad \mathcal{S}C\mathcal{S}^{-1} = H.$$

5. PROOF OF THEOREM 1.4

We note that \mathcal{T}_q and the center of $\overline{\mathcal{U}}_q sl(2)$ from [6] have the same dimension equal to $3p-1$. Then we identify C -basis (H -basis) in \mathcal{T}_q with the Radford images (Drinfeld images) of q -characters of irreducible representations

$$(5.1) \quad \begin{aligned} \widehat{\boldsymbol{\phi}}^+(s) &= \omega_s \mathbf{w}_s^+, \quad \widehat{\boldsymbol{\phi}}^-(s) = \omega_{p-s} \mathbf{w}_{p-s}^-, \quad s = 1 \dots p-1, \\ \widehat{\boldsymbol{\phi}}^+(p) &= p\sqrt{2p} \mathbf{e}_p, \quad \widehat{\boldsymbol{\phi}}^-(p) = (-1)^{p+1} p\sqrt{2p} \mathbf{e}_0, \\ \boldsymbol{\chi}^+(s) &= \omega_s \mathbf{u}_s^+, \quad \boldsymbol{\chi}^-(s) = \omega_{p-s} \mathbf{u}_{p-s}^-, \quad s = 1 \dots p-1, \\ \boldsymbol{\chi}^+(p) &= p\sqrt{2p} \mathbf{f}_p, \quad \boldsymbol{\chi}^-(p) = (-1)^{p+1} p\sqrt{2p} \mathbf{f}_0. \end{aligned}$$

This identification establishes an isomorphism between \mathcal{T}_q and the center of $\overline{\mathcal{U}}_q sl(2)$ as associative commutative algebras.

Under the identification (5.1), \mathcal{T}_q coincides with the center of $\overline{\mathcal{U}}_q sl(2)$ as the representation of $SL(2, \mathbb{Z})$. In particular, the relations $\mathcal{S}(\boldsymbol{\chi}^\pm(s)) = \widehat{\boldsymbol{\phi}}^\pm(s)$ for $s = 0 \dots p$ in the center are parallel to the relations (4.10) in \mathcal{T}_q . The Gaussian element \mathbf{v} in notations of [6]

$$(5.2) \quad \mathbf{v} = \sum_{s=0}^p (-1)^{s+1} q^{-\frac{1}{2}(s^2-1)} \mathbf{e}_s + \sum_{s=1}^{p-1} (-1)^p q^{-\frac{1}{2}(s^2-1)} \frac{q^s - q^{-s}}{\sqrt{2p}} \widehat{\boldsymbol{\varphi}}(s),$$

where $\widehat{\boldsymbol{\varphi}}(s) = \frac{p-s}{p} \widehat{\boldsymbol{\phi}}^+(s) - \frac{s}{p} \widehat{\boldsymbol{\phi}}^-(p-s)$ for $1 \leq s \leq p-1$ coincides with the ribbon element of $\overline{\mathcal{U}}_q sl(2)$.

6. DISCUSSION

We identified the representation of DAHA that gives the Verlinde algebra of $(1, p)$ logarithmic conformal field models. The center of $\overline{\mathcal{U}}_q sl(2)$ coincides with the symmetrization of \mathcal{Z} and $C = -(X + X^{-1})$ coincides with the $\overline{\mathcal{U}}_q sl(2)$ Casimir element. Probably the

whole representation \mathcal{Z} can be realized in $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ such that X would be realized by a multiplication with a $\overline{\mathcal{U}}_{\mathfrak{q}}sl(2)$ element.

Another interesting direction of investigations is to find a realization of \mathcal{H} on $(1, p)$ logarithmic conformal field model conformal blocks. This can also be useful in boundary conformal field theories. The Ishibashi and Cardy boundary states can probably be identified with eigenvectors of operators $C = -(X + X^{-1})$ and $H = -(Y + Y^{-1})$ respectively.

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APPENDIX A. PROOF OF $\mathcal{S}^2 = \mathfrak{q}T^{-1}$

We calculate coefficient in front of e_j in A.1 and coefficient in front of w_j in A.2.

A.1. The coefficient in front of e_j . The substitution of (2.26), (2.28), (2.30) in (3.5) gives the coefficient in front of e_j

$$(A.1) \quad \begin{aligned} & f_{1,s}^{(w)} u_{j,1}^{(e)} + f_{p,s}^{(w)} u_{j,p}^{(e)} + f_{p+1,s}^{(w)} u_{j,p+1}^{(e)} + f_{2p,s}^{(w)} u_{j,2p}^{(e)} + f_{p,s}^{(e)} f_{j,p}^{(e)} + f_{2p,s}^{(e)} f_{j,2p}^{(e)} + \\ & + \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \underbrace{\left(f_{r,s}^{(w)} u_{j,r}^{(e)} + f_{r,s}^{(m)} k_{j,r}^{(e)} \right)}_A, \end{aligned}$$

where all numbers u, k, f are given in (2.27), (2.29), (2.31). A simplification of the underbraced expression gives

for $j \neq 1, p, p+1, 2p$:

$$\begin{aligned} A = & \frac{(-1)^{s+j} \mathfrak{q}^2}{2p} \left(\mathfrak{q}^{r-1} [s, r-1] [r, j] - [s, r-1] [r, j-1] \right) + \\ & + \frac{(-1)^{s+j} \mathfrak{q}^s}{2p} \left(\mathfrak{q} [s, r] [r, j-1] - \mathfrak{q}^r [s, r] [r, j] \right), \end{aligned}$$

for $j = 1$:

$$A = \frac{(-1)^{s+1} \mathfrak{q}^2}{2p} \left(\mathfrak{q}^{r-1} [s, r-1] [r, 1] - [s, r-1] \right) + \frac{(-1)^{s+1} \mathfrak{q}^s}{2p} \left(-\mathfrak{q} \{s, r\} - \mathfrak{q}^r [s, r] [r, 1] \right),$$

for $j = p+1$:

$$A = \frac{(-1)^{s+p+1} \mathfrak{q}^2}{2p} \left(\mathfrak{q}^{r-1} [s, r-1] [r, p+1] - [s+p, r-1] \right) +$$

$$+ \frac{(-1)^{s+p+1} \mathbf{q}^s}{2p} \left(\mathbf{q}\{s+p, r\} - \mathbf{q}^r[s, r][r, p+1] \right),$$

for $j = p$:

$$A = \frac{(-1)^{s+p} \mathbf{q}^2}{2p} \left(\mathbf{q}^{r-1}[s+p, r-1] - [s, r-1][r, p-1] \right) + \\ + \frac{(-1)^{s+p} \mathbf{q}^s}{2p} \left(\mathbf{q}[s, r][r, p-1] - \mathbf{q}^r\{s+p, r\} \right),$$

for $j = 2p$:

$$A = \frac{(-1)^s \mathbf{q}^2}{2p} \left(\mathbf{q}^{r-1}[s, r-1] - [s, r-1][r, 2p-1] \right) + \frac{(-1)^s \mathbf{q}^s}{2p} \left(\mathbf{q}[s, r][r, 2p-1] + \mathbf{q}^r\{s, r\} \right).$$

Then the simplification of (A.1) gives coefficients in (3.5) in front of e_s . Explicitly, the summation in r of different terms in A is given by

$$\left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \mathbf{q}^{r-1}[s, r-1][r, j] = \left(p - \frac{1}{4}((s+j+1 \bmod 2p) + (s-j+1 \bmod 2p) + \right. \\ \left. + (s+j-1 \bmod 2p) + (s-j-1 \bmod 2p)) \right) \left(1 + (-1)^{s+j} \right) + p \frac{\{j\}}{[j]} (\delta_{s+j, 2p} - \delta_{s-j, 0}),$$

$$\left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) [s, r-1][r, j] = \left(1 + (-1)^{s+j+1} \right) \left(p - \frac{1}{2}((s+j \bmod 2p) + \right. \\ \left. + (s-j \bmod 2p)) \right)$$

$$\left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) [s, r][r, j] = [s]((-1)^{s+j} - 1), \quad j \neq p, 2p$$

$$\left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \mathbf{q}^r[s, r][r, j] = \frac{2p(\delta_{s, 2p-j} - \delta_{s, j})}{\mathbf{q}^j - \mathbf{q}^{-j}} - \mathbf{q}[s](1 + (-1)^{j+s}), \quad j \neq p, 2p,$$

$$\left(\sum_{r=1}^{p-1} + \sum_{r=p+1}^{2p-1} \right) [s, r] = (1 - (-1)^s)(p - (s \bmod 2p)),$$

$$\sum_{r=1}^{2p} \{s, r\} = 0,$$

$$\left(\sum_{r=1}^{p-1} + \sum_{r=p+1}^{2p-1} \right) \mathbf{q}^r[s, r] = (1 + (-1)^s) \left(p - \frac{((s+1) \bmod 2p) + ((s-1) \bmod 2p)}{2} \right),$$

$$\sum_{r=1}^{2p} \mathbf{q}^r\{s, r\} = (1 + (-1)^s) \left(\frac{((s-1) \bmod 2p) - ((s+1) \bmod 2p)}{2} \right).$$

A.2. The coefficient in front of w_j . The substitution of (2.26), (2.28), (2.30) in (3.5) gives the coefficient in front of w_j

$$(A.2) \quad f_{1,s}^{(w)} u_{j,1}^{(w)} + f_{p,s}^{(w)} u_{j,p}^{(w)} + f_{p+1,s}^{(w)} u_{j,p+1}^{(w)} + f_{2p,s}^{(w)} u_{j,2p}^{(w)} + f_{p,s}^{(e)} f_{j,p}^{(w)} + f_{2p,s}^{(e)} f_{j,2p}^{(w)} + \\ + \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \underbrace{(f_{r,s}^{(w)} u_{j,r}^{(w)} + f_{r,s}^{(m)} k_{j,r}^{(w)})}_A,$$

where all numbers u , k , f are given in (2.27), (2.29), (2.31). A simplification of the underbraced expression gives

$$(A.3) \quad A = \frac{q^2(-1)^{s+j}\{1, j-1\}}{p(q^{j-1} + q^{-j+1})} [s, r-1][r-1, j-1] - \frac{(-1)^{s+j}\{1, j\}q^2}{p(q^j + q^{-j})} q^{r-1} [s, r-1][r-1, j] + \\ + \frac{q^s(-1)^{s+j}}{2p} \left(q^r \{s, r\} \{r, j\} - q \{s, r\} \{r, j-1\} + q^r [s, r][r, j] \{1, j\} - q [s, r][r, j-1] \{1, j-1\} \right)$$

Then the simplification of (A.2) gives coefficients in (3.5) in front of w_s . Explicitly, the summation in r of different terms in (A.3) is given by

$$\left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) [s, r-1][r-1, j] = [s](1 - (-1)^{s+j}), \quad j \neq p, 2p, \\ \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q^{r-1} [s, r-1][r-1, j] = \frac{2p(\delta_{s,2p-j} - \delta_{s,j})}{q^j - q^{-j}} + \\ + q^{-1} [s](1 + (-1)^{j+s}), \quad j \neq p, 2p, \\ \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q^r \{s, r\} \{r, j\} = \left(\frac{p(\delta_{s+j,2p} + \delta_{s-j,0}) - 2}{q^j + q^{-j}} - q \{s\} \right) \{1, j\} (1 + (-1)^{j+s}), \\ \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) \{s, r\} \{r, j\} = \{s, 1\} \{1, j\} ((-1)^{s+j} - 1), \\ \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) q^r [s, r][r, j] = \frac{2p(\delta_{s,2p-j} - \delta_{s,j})}{q^j - q^{-j}} - q [s](1 + (-1)^{j+s}), \quad j \neq p, 2p, \\ \left(\sum_{r=2}^{p-1} + \sum_{r=p+2}^{2p-1} \right) [s, r][r, j] = [s]((-1)^{s+j} - 1), \quad j \neq p, 2p.$$

REFERENCES

- [1] A.M. Semikhatov, *Factorizable ribbon quantum groups in logarithmic conformal field theories*, [hep-th/0705.4267].
- [2] V. Gurarie, *Logarithmic operators in conformal field theory*, Nucl. Phys. B410 (1993) 535 [hep-th/9303160].

- [3] M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. B386 (1996) 131–137 [hep-th/9606050].
- [4] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet W -algebra in logarithmic CFT*, math.QA/0512621.
- [5] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, I, J. Amer. Math. Soc. 6 (1993) 905–947; II, J. Amer. Math. Soc. 6 (1993) 949–1011; III, J. Amer. Math. Soc. 7 (1994) 335–381; IV, J. Amer. Math. Soc. 7 (1994) 383–453.
- [6] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun. Math. Phys. 265 (2006) 47–93 [hep-th/0504093].
- [7] M.R. Gaberdiel and H.G. Kausch, *Indecomposable fusion products*, Nucl. Phys. B477 (1996) 293–318 [hep-th/9604026].
- [8] J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. 247 (2004) 713–742 [hep-th/0306274].
- [9] M. Flohr and H. Knuth, *On Verlinde-Like Formulas in $c_{p,1}$ Logarithmic Conformal Field Theories*, math-ph/0705.0545.
- [10] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, Int. J. Mod. Phys. A18 (2003) 4497–4592 [hep-th/0111228].
- [11] I. Cherednik, *Double Affine Hecke Algebras*, 2004.
- [12] I. Cherednik, Private communication, Kyoto, 2004.

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